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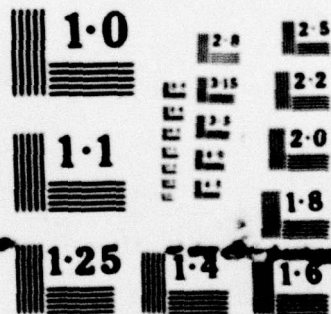UNCLASSIFIED

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This paper contains an extension to C^k-vector fields of the classical results of Liapunov on the stability of an equilibrium point in the critical case of one zero root. The transformation theory of Liapunov is not applicable to this case. We exploit fundamental relations from bifurcation theory.

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STABILITY FROM THE BIFURCATION FUNCTION[*]

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STABILITY FROM THE BIFURCATION FUNCTION

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ABSTRACT

This paper contains an extension to C^k-vector fields
of the classical results of Liapunov on the stability
of an equilibrium point in the critical case of one zero
root. The transformation theory of Liapunov is not
applicable to this case. We exploit fundamental relations
from bifurcation theory.

In a recent paper, deOliveira and Hale [2] discussed the relationship between the bifurcation function obtained by the method of Liapunov-Schmidt and the flow on the center manifold for periodic n-dimensional systems for which the linear approximation has one characteristic multiplier equal to one and the remaining ones inside the unit circle. The purpose of this paper is to present a more elementary proof of this result and also to give an application to stability for autonomous systems in the critical case of one zero root. This allows one to extend and simplify the classical results of Liapunov [4] to the case of C^k-vector fields. The approach used by Liapunov cannot be extended to this case since he employs the theory of transformations to approximate the vector fields by simpler vector fields.

Consider the n-dimensional vector equation

(1) $$\dot{x} = Ax + F(t,x,\lambda), \qquad A = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

where $\lambda \in E$, a Banach space, B is an $(n-1) \times (n-1)$ matrix with eigenvalues with negative real parts, $F(t,x,\lambda)$ is continuous, is C^k, $k \geq 1$, in x, λ, $F(t+1,x,\lambda) = F(t,x,\lambda), F(t,0,0) = 0$, $\partial F(t,0,0)/\partial x = 0$. Our objective is to determine the 1-periodic solutions of (1) in a neighborhood of $(x,\lambda) = (0,0)$ as well as their stability properties.

The method of Liapunov-Schmidt is an effective way of determining the number of 1-periodic solutions of (1). Since A has

0 as a simple eigenvalue and all other eigenvalues with
negative real parts, this method yields a scalar function
$G(a,\lambda)$ defined and C^k for (a,λ) in a neighborhood of
$(0,0) \in \mathbb{R} \times E$ such that Eq. (1) has a 1-periodic solution
in a certain neighborhood of $(x,\lambda) = (0,0)$ if and only if
$G(a,\lambda) = 0$. Also, there is a prescription for determining
the 1-periodic solutions from the zeros of G. The method
of Liapunov-Schmidt can be considered as a reduction principle
- reducing the discussion of the existence of special solutions
of (1) (the 1-periodic solutions) to an alternative problem in
lower dimension - namely, to finding the zeros of the scalar
function $G(a,\lambda)$.

There is another reduction principle in differential equa-
tions which determines the behavior of all solutions of (1) in
a neighborhood of $(x,\lambda) = 0$. This is the center manifold
theorem. If $x = (y,z)$, $y \in \mathbb{R}$, $z \in \mathbb{R}^{n-1}$ in Eq. (1), this
theorem says there is a function $h(t,y,\lambda) \in \mathbb{R}^{n-1}$, defined,
continuous, C^k in y,λ in $\mathbb{R} \times U$, where U is a neighbor-
hood of $(y,\lambda) = (0,0) \in \mathbb{R} \times E$, such that $h(t,y,\lambda) = h(t+1,y,\lambda)$
and the set

$$M_\lambda \overset{\text{def}}{=} \{(t,y,z):y \in \mathbb{R}, z \in \mathbb{R}^{n-1}, z = h(t,y,\lambda), t \in \mathbb{R}, (y,\lambda) \in U\}$$

is an invariant manifold for Eq. (1). Furthermore, this manifold
M_λ is exponentially asymptotically stable. The properties of

the solutions of Eq. (1) in a neighborhood of $(x,\lambda) = (0,0)$ is completely determined by the flow induced on M_λ by the scalar equation

$$(2) \qquad \dot{y} = f(t,y,h(t,y,\lambda),\lambda)$$

where $F = (f,g)$, $f \in \mathbb{R}$, $g \in \mathbb{R}^{n-1}$. Any 1-periodic solution of Eq. (1) near $(x,\lambda) = (0,0)$ is given by $x(t) = (y(t),z(t))$, $z(t) = h(t,y(t),\lambda)$, and $y(t)$ a 1-periodic solution of (2), and conversely. Furthermore, the stability properties of $y(t)$ determine the stability properties of $x(t)$ in the sense that they are the same except that $x(t)$ always has an additional $(n-1)$-dimensional stable manifold.

The principle result proved by deOliveira and Hale [2] is

Theorem 1. Consider the scalar equation

$$(3) \qquad \dot{a} = G(a,\lambda)$$

where $G(a,\lambda)$ is the bifurcation function obtained by the method of Liapunov-Schmidt. The zeros of $G(a,\lambda)$ are in one-to-one correspondence with the 1-periodic solutions of Eq. (2) in a neighborhood of $(y,\lambda) = (0,0)$. Furthermore, the stability properties of the equilibrium points of (3) are the same as the stability properties of the 1-periodic solutions of (2).

Proof: Let us indicate a proof which is different from and simpler than the one in [2]. We first prove it for the scalar equation (2). For Eq. (2), one can also apply the method of Liapunov-Schmidt to obtain a scalar bifurcation $\tilde{G}(a,\lambda)$ in the following way. In a neighborhood of $(y,a,\lambda) = (0,0,0)$, let $y(t,a,\lambda)$ be the unique 1-periodic solution of the equation

$$(4) \quad \dot{y}(t) = f(t,y(t),h(t,y(t),\lambda)) - \int_0^1 f(s,y(s),h(\dot{s},y(s),\lambda),\lambda)ds$$

with $\int_0^1 y = a$ and define

$$(5) \qquad \tilde{G}(a,\lambda) = \int_0^1 f(s,y(s,a,\lambda),h(s,y(s,a,\lambda),\lambda),\lambda)ds.$$

The 1-periodic solutions of Eq. (2) in a neighborhood of zero are in one-to-one correspondence with the zeros of $\tilde{G}$ in a neighborhood of zero. Let us make the transformation of variables

$$y \mapsto b : y(t) = y(t,b,\lambda)$$

in Eq. (2). Since $y(t,b,0) = b$, $\partial y(t,0,0)/\partial b = 1$, and $y(t,b,\lambda)$ satisfies (4), (5), we have

$$(6) \qquad \dot{b} = (\partial y(t,b,\lambda)/\partial b)^{-1}\tilde{G}(b,\lambda).$$

For (b,λ) in a sufficiently small neighborhood of zero,

$(\partial\gamma(t,b,\lambda)/\partial b)^{-1} > 0$. This immediately shows that the stability properties of any 1-periodic solution of (6) (which coincides with the zeros of $\tilde{G}$) is the same as the stability properties of the corresponding solution of $\dot{a} = \tilde{G}(a,\lambda)$. This proves the theorem for Eq. (2).

To prove the theorem for Eq. (1), we proceed as in [2], first observing that $\tilde{G}(a,\lambda), G(a,\lambda)$ have the same set of zeros in a neighborhood of zero. Thus, we need only show that $\tilde{G}, G$ have the same sign between zeros. Suppose this is not the case. By a small perturbation in the original Eq. (1), we can suppose the zeros of both functions are simple. Now replace f in Eq. (1) by $f + \epsilon$, obtain the new bifurcation $G(a,\lambda,\epsilon) = G(a,\lambda) + \epsilon$ for (1), obtain the new bifurcation function $\tilde{G}(a,\lambda,\epsilon) = \tilde{G}(a,\lambda) + \delta(a,\lambda)\epsilon + O(|\epsilon|^2)$ as $|\epsilon| \to 0$, $\delta(a,\lambda) > 0$, for the corresponding equation on the center manifold. Now suppose (a_0,λ_0) is a simple zero of $G(a,\lambda), \tilde{G}(a,\lambda)$ with $(\partial G/\partial a)(\delta\tilde{G}/\delta a) < 0$ at (a_0,λ_0); that is, the functions have opposite sign in a neighborhood of (a_0,λ_0). The functions $G(a,\lambda,\epsilon), \tilde{G}(a,\lambda,\epsilon)$ have distinct zeros in a neighborhood of (a_0,λ_0) for ϵ small. But this is a contradiction since they must have the same zeros. This proves the theorem.

Applications of Theorem 1 to general bifurcation theory and to Hopf bifurcation may be found in [2].

Also, the extension to infinite dimensional systems is discussed - especially parabolic equations and functional differential equations.

Let us give another application to the classical problem of stability in critical cases discussed so thoroughly by Liapunov [4], Lefschetz [3] and, more recently, by Bibikov [1] as well as others.

Consider the equation

$$\dot{y} = f(y,z)$$

(7)

$$\dot{z} = Bz + g(y,z)$$

with $y \in \mathbb{R}, z \in \mathbb{R}^{n-1}$, B an $(n-1) \times (n-1)$ matrix whose eigenvalues have negative real parts, f,g are C^k functions vanishing together with their first derivatives at $(0,0)$.

The method of Liapunov-Schmidt applied to this equation yields the bifurcation function

(8) $G(a) = f(a,\phi(a))$

where ϕ is the unique solution of the equation

(9) $B\phi + g(a,\phi) = 0$

in a neighborhood of $(a,\phi) = (0,0)$.

As immediate consequences of Theorem 1, we have

<u>Corollary 1</u>.   <u>If there is an integer</u> $q \geq 2$ <u>and</u> $\beta \neq 0$ <u>such that</u>

$$G(a) = \beta a^q + O(|a|^{q+1}) \quad \text{as} \quad |a| \to 0$$

<u>then the solution</u> $x = (y,z)$ <u>of</u> (7) <u>is asymptotically stable if and only if</u> $\beta < 0, q$ <u>odd. Otherwise, it is unstable</u>.

This is the classical result of Liapunov. The function $G(a)$ can be C^∞ and there may never exist a β, q as in Corollary 1. Theorem 1 actually implies a more general property.

<u>Corollary 2</u>.   <u>The solution</u> $(y,z) = (0,0)$ <u>of</u> (7) <u>is asymptotically stable if and only if there is an</u> $\epsilon > 0$ <u>such that</u> $G(a)a < 0$ <u>for</u> $0 < |a| < \epsilon$. <u>The solution</u> $(y,z) = (0,0)$ <u>is unstable if and only if there is an</u> $\epsilon > 0$ <u>such that</u> $G(a)a > 0$ <u>for either</u> $0 < a < \epsilon$ <u>or</u> $-\epsilon < a < 0$. <u>If there is an</u> $\epsilon > 0$ <u>such that</u> $G(a) = 0$ <u>for</u> $0 \leq |a| < \epsilon$, <u>then the solution</u> $(y,z) = (0,0)$ <u>of</u> (7) <u>is stable and there is a first integral in a neighborhood of</u> $(0,0)$.

<u>Proof</u>: Everything is obvious from Theorem 1 except the existence of the first integral. Suppose $G(a) = 0$ for $|a| < \epsilon$ and let

$z = h(y)$ be the parametric representation of a center manifold M at $(0,0)$: $M = \{(y,z): z = h(y)\}$. We know that $f(y,h(y)) = 0$ for $|y| < \epsilon$. Also, each equilibrium point $(y_0,h(y_0))$ of (7) $|y_0| < \epsilon$, has an $(n-1)$-dimensional stable manifold $S(y_0)$ which is C^k in y_0. The curve M and these stable manifolds can be used as a coordinate system in a neighborhood of $(0,0)$ to obtain a C^k mapping $H: (y,z) \mapsto (u,v)$, where $\dot{u} = 0$, $\dot{v} = Bv + \tilde{g}(u,v)$. This latter equation has a first integral $V(u,v) = u$ so that the original equation has a first integral. This proves the result.

Corollary 3. If there is scalar function $H(y,z,w)$, continuous for $(y,z,w) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ such that $H(y,z,0) = 0$ and

$$f(y,z) = H(y,z,Bz + g(y,z))$$

then the zero solution of (1) is stable and there is a first integral.

Proof: The hypotheses imply $G(y) = f(y,\phi(y)) = 0$ for y in a neighborhood of zero and so Corollary 2 applies.

In the case of analytic systems, Bibikov [1] refers to the situation in Corollary 1 as the algebraic case. For f,g analytic, the function $G(a)$ is analytic and therefore either the algebraic case holds or $G(a) \equiv 0$, which is called the

transcendental case. Corollary 2 says there is a first integral
in the analytic case - another classical result of Liapunov
(see Bibikov [1]). Corollary 3 was also stated by Liapunov
for analytic systems. Thus, we see that the basic results of
Liapunov can be generalized to C^k-vector fields and, in
addition, everything is based only on the bifurcation function.
This latter remark is the essential improvement in the state-
ment of the results of Liapunov. Some aspects of the proofs
however are similar. Liapunov used his general transformation
theory to put the equation in a form where it is easy to dis-
cover the center manifold and the flow on the center manifold.
We use the abstract center manifold theory and properties of
the stable manifold. In addition, a small amount of perturbation
theory is used in an abstract way to prove the bifurcation
function determines the stability properties of the solutions.

It is instructive for the reader to check the original
examples given by Liapunov [4] to see how only the bifurcation
function was used to determine stability.

We remark that the same results as above have extensions to
certain evolutionary equations in infinite dimensions; for
example, parabolic systems and functional differential equations.

Finally, we have emphasized stability of the solution
$(y,z) = 0$ of (7). If the autonomous system depends on a para-
meter (which often occurs in applications), Theorem 1 may be
applied directly to obtain stability results even at the bifurca-
tion curves.

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